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HW 7

MATH 527B
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1. (M, d) is a metric space. As defined in class, let $\bar{M}$ be the space of equivalence classes of Cauchy sequences, with

$$\bar{d}([\{x_n\}], [\{y_n\}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Also, $\phi : M \to \bar{M}$ denotes the canonical map $x \mapsto [\{x, x, x, \dots\}]$, that maps each point in M to the equivalence class of the constant sequence of this element.

(a) Show that $\phi(M)$ is dense in $\bar{M}$, *i.e.*, given any equivalence class of Cauchy sequences $[\{x_n\}] \in \bar{M}$ and an $\epsilon > 0$, there is a point $y \in M$ such that $\bar{d}([\{x_n\}], \phi(y)) < \epsilon$.

Solution. Pick some $[\{x_n\}] \in \bar{M}$ and some $\epsilon > 0$. Now, by definition

$$\begin{aligned}
\bar{d}([\{x_n\}], \phi(y)) &= \bar{d}([\{x_n\}], [\{y\}]) \\
&= \lim_{n\to\infty} d(x_n, y) \\
&\leq \lim_{n\to\infty} \big(d(x_n, x_m) + d(x_m, y)\big) \\
&= \lim_{n\to\infty} d(x_n, x_m) + d(x_m, y)
\end{aligned}$$

For some index $m \in \mathbb{N}$. Now, we know that the sequence $\{x_n\}$ is Cauchy, therefore take ϵ same as in original assumption. $\exists N$ s.t. $m, n > N \implies d(x_m, x_n) < \frac{\epsilon}{2}$. Take m above to be $> N$. Therefore the above inequality becomes

$$\begin{aligned}
\bar{d}([\{x_n\}], \phi(y)) &\leq \lim_{n\to\infty} d(x_n, x_m) + d(x_m, y) \\
&< \frac{\epsilon}{2} + d(x_m, y) \\
&< \epsilon.
\end{aligned}$$

The last inequality is true since I could pick $y = x_m$ for any index m, since $y = x_m \in M$. This shows the desired result.

(b) Show that $(\bar{M}, \bar{d})$ is a complete metric space.

Solution. Define a function $f : M \to \bar{M}$ by

$$f(x) = [\{x\}],$$

where $[\{x\}]$ denotes the equivalence class whose members converge to x. For example then, the sequence $\{x, x, \dots, x\} \in [\{x\}]$. Hence f is an injection. Now define $\bar{d}$ as above, where $x_n \in [\{x\}]$, and $y_n \in [\{y\}]$.

Proposition. *Given $\{x_n\} \in [\{x\}]$ and $\{y_n\} \in [\{y\}]$, then $\{d(x_n, y_n)\}$ is a Cauchy sequence, whose limit exists in $\mathbb{R}$.*

Proof. Assume $\{x_n\} \in [\{x\}]$ and $\{y_n\} \in [\{y\}]$. Then there exists an N_1 such that $n, m \geq N_1$ implies that $d(x_n, x_m) < \epsilon/2$, and an N_2 such that $n, m \geq N_2$ implies that $d(y_n, y_m) < \epsilon/2$. Set $N = \max(N_1, N_2)$, then for all $m, n \geq N$ we have that

$$\begin{aligned}
|d(x_n, y_n) - d(x_m, y_m)| &\leq |d(x_n, x_m) + d(x_m, y_n) - d(y_m, y_n) - d(y_n, y_m)| \\
&\leq d(x_n, x_m) + d(y_n, y_m) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Hence, since $\{d(x_n, y_n)\}$ is Cauchy, then $\lim_{n\to\infty} d(x_n, y_n)$ exists, since $\mathbb{R}$ is complete.

♣

Next, we show that $\bar{d}$ is a metric on $\bar{M}$. Since

$$\begin{aligned}
d(x_n, y_n) \geq 0 &\Rightarrow \lim_{n\to\infty} d(x_n, y_n) \geq 0, \quad \text{hence} \\
&\Rightarrow \bar{d}([\{x\}], [\{y\}]) \geq 0
\end{aligned}$$

Now suppose that $[\{x\}] = [\{y\}]$, then $\bar{d}([\{x\}], [\{y\}]) = \lim_{n\to\infty} d(x_n, y_n)$ where $\{x_n\} \in [\{x\}]$ and $\{y_n\} \in [\{y\}]$. But, since $\{x_n\} \sim \{y_n\}$, thus

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, y_n) = 0, &\quad \text{since the both have the same limit.} \\
\Rightarrow \bar{d}([\{x\}], [\{y\}]) = 0&
\end{aligned}$$

Conversely, let $\bar{d}([\{x\}], [\{y\}]) = 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$, which then implies that $\{x_n\} \sim \{y_n\}$, for $\{x_n\} \in [\{x\}]$, $\{y_n\} \in [\{y\}]$. Hence $[\{x\}] = [\{y\}]$. We have symmetry of $\bar{d}$ automatically, by the symmetry in d. Finally, we must show the triangle inequality. then for $\{x_n\} \in [\{x\}]$, $\{y_n\} \in [\{y\}]$ and $\{z_n\} \in [\{z\}]$ we have

$$\begin{aligned}
\bar{d}([\{x\}], [\{z\}]) &= \lim_{n\to\infty} d(x_n, z_n) \\
&\leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\
&= \bar{d}([\{x\}], [\{y\}]) + \bar{d}([\{y\}], [\{z\}])
\end{aligned}$$

Now, we note that $\bar{d}([\{x\}], [\{y\}]) = \bar{d}(f(x), f(y)) = d(x, y)$, for all $x, y \in M$ by construction. Hence f is an *isometry* from $M \to \bar{M}$. Now we must show that every $[\{x\}] \in \bar{M}$ is the limit of a sequence in $f(M)$. Suppose that the sequence $\{x_n\}$

were Cauchy, and in $[\{x\}]$. Then for $k \in \mathbb{N}$, $\exists N_k$ such that $n \geq N_k$ implies that $d(x_n, x_{N_k}) < \frac{1}{k}$. Define $[\{y_k\}]$ so that

$$[\{y_k\}] = f(x_{N_k})$$

which is the equivalence class of all Cauchy Sequences converging to x_{N_k}. Thus

$$\begin{aligned}\bar{d}([\{x\}], [\{y_k\}] &= \bar{d}([\{x\}], f(x_{N_k})) \\ &= \lim_{n\to\infty} d(x_n, x_{N_k}) \\ &\leq \frac{1}{k}\end{aligned}$$

Therefore we have that $\lim_{k\to\infty} f(x_{N_k}) = [\{x\}]$.

Finally, we show that $\bar{M}$ is complete. Let $\{[\{x\}]^n\}$ be Cauchy in $\bar{M}$. Given that each $[\{x\}]^n$ is the limit of a sequence in $f(M)$, we may find, for sufficiently large n, $[\{y\}]^n \in f(M)$ such that

$$\bar{d}([\{x\}]^n, [\{y\}]^n) < \frac{1}{n}$$

We can show that $\{[\{y\}]^n\}$ is a Cauchy sequence in $\bar{M}$, since for all $\epsilon > 0$, there exists $N_1(\epsilon)$ such that for $n \geq N_1(\epsilon) \Rightarrow \frac{1}{n} < \epsilon/3$. Also, there exists some $N_2(\epsilon)$ such that $m \geq N_2(\epsilon) \Rightarrow \frac{1}{m} < \epsilon/3$, and also $\exists N_3(\epsilon)$ such that $m, n \geq N_3(\epsilon) \Rightarrow \bar{d}([\{x\}]^n, [\{x\}]^m) < \epsilon/3$. Then setting $N = \max(N_1(\epsilon), N_2(\epsilon), N_3(\epsilon))$. Thus

$$\begin{aligned}\bar{d}([\{y\}]^n, [\{y\}]^m) &\leq \bar{d}([\{y\}]^n, [\{x\}]^n) + \bar{d}([\{x\}]^n, [\{x\}]^m) + \bar{d}([\{x\}]^m, [\{y\}]^m) \\ &\leq \frac{1}{n} + \bar{d}([\{x\}]^n, [\{x\}]^m) + \frac{1}{m} \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

Now, since $[\{y\}]^n \in f(M)$, then there exists $y_n \in M$ such that $f(y_n) = [\{y\}]^n$, which implies that $\{y_n\}$ must be Cauchy in M, given that $[\{y\}]^n$ is Cauchy in $\bar{M}$ (*using the fact that f is an isometry*). Hence $\{y_n\}$ converges in M, and so it belongs to some equivalence class, say $[\{x\}] \in \bar{M}$. Now I show

$$\lim_{n\to\infty} \bar{d}([\{x\}]^n, [\{x\}]) = 0$$

Let $\epsilon > 0$, then there exists N_ϵ so that $n \geq N_\epsilon$ guarantees that $\frac{1}{n} < \epsilon/2$, as well as $d(y_n, x) < \epsilon/2$ (*Recall that $y_n \sim x$*). Thus

$$
\begin{aligned}
\bar{d}([\{x\}]^n, [\{x\}]) &\leq \bar{d}([\{x\}]^n, [\{y\}]^n) + \bar{d}([\{y\}]^n, [\{x\}]) \\
&\leq \frac{1}{n} + \bar{d}(f(y_n), [\{x\}]) \\
&= \frac{1}{n} + \lim_{n\to\infty} d(y_n, x) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

Hence we have that $\lim_{n\to\infty} \bar{d}([\{x\}]^n, [\{x\}]) = 0$.

$\therefore \bar{M}$ is complete.

(c) (M, d) and (N, ρ) are metric spaces. A function $f : M \to N$ is uniformly continuous, if for all $\delta > 0$, there exists an $\epsilon > 0$ such that $d(x, y) < \epsilon \implies \rho(f(x), f(y)) < \delta$. Show that the map $\phi : (M, d) \to (\bar{M}, \bar{d})$ is uniformly continuous.

Solution. By definition,

$$
\begin{aligned}
\bar{d}(\phi(x), \phi(y)) &= \bar{d}([\{x\}], [\{y\}]) \\
&= \lim_{n\to\infty} d(x, y) = d(x, y)
\end{aligned}
$$

Therefore, $\forall\delta > 0$, there exists an $\epsilon = \delta > 0$ such that $d(x, y) < \epsilon \implies \bar{d}(\phi(x), \phi(y)) < \delta$.

$\therefore$ the map $\phi : (M, d) \to (\bar{M}, \bar{d})$ is uniformly continuous.

(d) If $f : M \to N$ is uniformly continuous, show that there exists a unique continuous function $g : \bar{M} \to \bar{N}$ such that f and g agree on M. Here $\bar{M}$ and $\bar{N}$ are respectively the completions of M and N.

Solution. Break the problem into three components (1) show that a function g exists and agrees with f on M,(2) show that g is continuous and (3) show that g is unique.

(1) If $\bar{x} \in M$, or $\exists m \in M$ s.t. $\bar{x} = \{x, x, \dots\}$. Pick g s.t. $g(\bar{x}) = \overline{f(x)} = \{f(x, f(x), \dots\} \in \bar{N}$. Clearly, these two functions agree on M.

(2) If $\bar{x} \notin M$, then $\exists$ a Cauchy sequence $\{x_n\} \in M$ s.t. $\bar{x} = [\{x_n\}]$. Now, as defined $g(\bar{x}) = [\{f(x_1), f(x_2), \dots, f(x_n)\}]$. Putting together the facts that $\{x_n\}$ is Cauchy and f is uniformly continuous implies that $\{f(x_n)\}$ is Cauchy. Therefore, $g(\bar{x}) \in \bar{N}$.

Suppose now that two sequences $\bar{x}, \bar{y}$ are in $\bar{M}$. Consider the quantity $\bar{d}(\bar{x}, \bar{y}) = \lim_{n\to\infty} d(x_n, y_n)$. From above paragraph, $g(\bar{x}), g(\bar{y}) \in \bar{N}$, $\bar{\rho}(g(\bar{x}), g(\bar{y})) = \lim_{n\to\infty} \rho(f(x_n), f(y_n))$. Taking into account that f is uniformly continuous implies g is continuous.

(3) Suppose two functions g_1, g_2 both satisfy the given conditions. Then $g = g_1 - g_2$ is continuous from $\bar{M} \to \bar{N}$. $g(\bar{x}) = 0$ if $\bar{x} \in M$. Since M is dense in $\bar{M}$, $g = 0$ in $\bar{M}$. So g must be the zero-function; i.e. $g_1 = g_2$. Therefore the function g from the problem definition is unique.

(e) Give an example of a continuous function $f : \mathbb{Q} \to \mathbb{Q}$ which has no extension as a continuous function from $\mathbb{R} \to \mathbb{R}$. What can you conclude from this example?

Solution. Take $f(x) = \frac{1}{x^2-\pi^2}$. This function is continuous from $\mathbb{Q} \to \mathbb{Q}$, since the zeroes of the polynomial $x^2 - \pi^2$ are $\notin \mathbb{Q}$. However, the extension to $\mathbb{R}$ has singularities at $\pm\pi$. Therefore, we can conclude that the function f is not uniformly continuous on $\mathbb{R}$.

2. (M, d) is a metric space. Show that the following properties are equivalent-

(a) $\exists K < \infty$ such that for all $x, y \in M, d(x, y) \leq K$.

(b) $\exists x_0 \in M$, $P < \infty$ such that $\forall y \in M$, $d(x_0, y) < P$.

A metric space which satisfies these characterizations is a *bounded* metric space.

Solution. First, show that (a) $\implies$ (b).

Assume $\exists K < \infty$ such that for all $x, y \in M, d(x, y) \leq K$. This direction is simple. Pick any value $x_0 \in M$. Then, the following is true for all $y \in M$

$$d(x_0, y) \leq K,$$

So, pick $P = K + 1$, therefore

$$d(x_0, y) < P.$$

which shows that (a) $\implies$ (b).

Now, assume $\exists x_0 \in M$, $P < \infty$ such that $\forall y \in M$, $d(x_0, y) < P$. Therefore, for any $x, y \in M$ we have that

$$d(x, y) \leq d(x, x_0) + d(x_0, y),$$

but we know that the right hand term of the inequality is $< 2P$. So, pick $P = \frac{K}{2}$, then

$$\begin{aligned} d(x,y) &\leq d(x,x_0) + d(x_0,y) \\ &< \frac{K}{2} + \frac{K}{2} \\ &\leq K. \end{aligned}$$

Therefore, (b) $\implies$ (a).

3. Completeness of $\mathbb{R}$.

(a) Assuming every bounded set in $\mathbb{R}$ has a smallest upper bound, show that $\mathbb{R}$ is a complete metric space.

Solution. Let A be a bounded set such that $A = \{x_1, x_2, \ldots, \}$, where $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. Given that A is bounded, it has a supremum, call it m. That is, if $x_i \in A$, then $x_i \leq m$. Then for all $n \in \mathbb{N}$, consider the set $L_1 = [m - \frac{1}{n}, m]$.

Should L_1 contain infinitely many $x_i \in \{x_n\}$, then we have that the limit of x of $\{x_n\}$ exists in L_1. Suppose then, that L_1 contains only finitely many points in $\{x_n\}$. Then let $B_1 = B \backslash L_1$. Thus B_1 is bounded, and we can find a sup of B_1, and continue the same process until we arrive at the desired result. Therefore we have that if every bounded set has a sup, then $\mathbb{R}$ is complete, since all Cauchy sequences in $\mathbb{R}$ have a limit which is in $\mathbb{R}$.

(b) Assuming that $\mathbb{R}$ is a complete metric space, show that every bounded set in $\mathbb{R}$ has a smallest upper bound.

Solution. Let $A \subseteq \mathbb{R}$, such that A is bounded. Hence there exists $\beta \in \mathbb{R}$ such that, $\forall \alpha \in A, \alpha \leq \beta$. We say β is an upper bound. Then

$$\begin{aligned} 0 \leq \beta - \alpha & \quad \text{and } \exists m, n \in \mathbb{N} \text{ such that} \\ \beta - \alpha \leq \frac{m}{n} & \\ \Downarrow & \\ \beta \leq \alpha + \frac{m}{n} & \end{aligned}$$

Hence $\alpha + \frac{m}{n}$ is an upper bound for A. Let $B_n = \{m \in \mathbb{N} : \alpha + \frac{m}{n} \text{ is an upper bound}\} \neq \emptyset$. Then for each n, there exists a smallest element $m_n \in B_n$ (*since* $m \in \mathbb{N}$) such that $\forall m \in B_n$, $m_n \leq m$. Therefore if we let

$$y_n = \alpha + \frac{m_n}{n}$$

And we have that y_n is an upper bound, for all $n \in \mathbb{N}$. Since m_n is the smallest natural number that makes y_n an upper bound, we have that

$$x_n = y_n - \frac{1}{n} = \alpha + \frac{m_n - 1}{n}$$

is not an upper bound. Hence there exists $x \in A$ such that $x_n < x < y_n$. Now, since $\mathbb{R}$ is complete, we have that the Cauchy sequence $\{x_n\}$ converges in $\mathbb{R}$, and equivalently, there exists some $\xi \in \mathbb{R}$ such that

$$\lim_{n \to \infty} x_n = \xi$$

Next, we show that this ξ is an upper bound. Supposing that ξ were not an upper bound, then there exists $x \in A$ such that $\xi < x$. Since $x_n \to \xi$, there exists some N_1 such that for every $\epsilon > 0$, $n \geq N_1$ guarantees that both $|x_n - \xi| < \frac{x-\epsilon}{2}$, and $\frac{1}{n} < \frac{x+\epsilon}{2}$. Thus we have

$$
\begin{aligned}
y_n &= x_n + \frac{1}{n} \\
&< \frac{x-\epsilon}{2} + \frac{x+\epsilon}{2} \\
&= x
\end{aligned}
$$

Implying that $y_n < x$, contradicting the fact that y_n is an upper bound of A. Therefore ξ is an upper bound. Finally, I show that ξ is the least upper bound of A. Again, suppose ξ were not the least upper bound. Hence there exists η such that $\eta < \xi$, and η is an upper bound of A. Letting $\delta = \xi - \eta > 0$, and using that $x_n \to \xi$, we have that there exists an N_δ such that $n \geq N_\delta$ implies

$$
\begin{aligned}
\xi - x_n &\leq |\xi - x_n| \\
&< \delta \\
&= \xi - \eta \\
&\Downarrow \\
\eta &< x_n
\end{aligned}
$$

Again, since there exists some $x \in A$ such that $x_n < x$, we have that η cannot be an upper bound of A. Therefore, ξ is the least upper bound.

$\therefore$ if $\mathbb{R}$ is complete, each bounded set in $\mathbb{R}$ has a smallest upper bound.

This shows that either of these statements can be taken as an axiom and the other statement follows.

4. Problem 1.2.15, pg. II-303.

   Show by example: two metrics d and d' may define the same topology on a set M, but (M, d) and (M', d') may fail to be isometric.

   Solution. Let $M = \{0, 1\}$. Let

$$
d_1(x,y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y. \end{cases}, \quad d_2(x,y) = \begin{cases} 1/2, & x \neq y; \\ 0, & x = y. \end{cases}
$$

   Then d_1 and d_2 define the same topology on M. Assume (M, d_1) and (M, d_2) are isometric. Then there exists an onto isometry $F : (M, d_1) \to (M, d_2)$. So we either have F_1, where $F_1(0) = 0$ and $F_1(1) = 1$, or F_2, where $F_2(0) = 1$ and $F_2(1) = 0$. If we have F_1, then

$$
d_2(F_1(0), F_1(1)) = d_2(0, 1) = 1/2 \neq 1 = d_1(0, 1).
$$

   So we have a contradiction. So assume we have F_2. Then

$$
d_2(F_2(0), F_2(1)) = d_2(1, 0) = 1/2 \neq 1 = d_1(1, 0).
$$

   Again we have a contradiction. Thus, (M, d_1) and (M, d_2) are not isometric.

5. Problem 1.2.16, pg. II-303.

   Prove Proposition 1.2.12:

   Proposition. *(1.2.12)*

   Let (M, d) be a metric space with completion $(\bar{M}, \bar{d})$. Suppose that (M', d') is a complete metric space, and suppose that there is an isometry $F : M \to M'$ whose range $F(M)$ is dense in M'. Then (M', d') and $(\bar{M}, \bar{d})$ are isometric.

   Proof. Let $\phi : \bar{M} \to M'$, such that

$$\phi([\{x_n\}]) = \lim_{n\to\infty} F(x_n), \quad x_n \in M.$$

   We then have that ϕ is well-defined, since M and M' are isometric, and $\bar{M}$ is the completion of M. Also, it is clear that ϕ is surjective, since $F(M)$ is dense in M', thus $\overline{F(M)} = M'$.

   Then, for every $x_n \in M$, there exists a sequence $\{x'_n\} \in M'$ such that for all i

$$F(x_i) = x'_i$$

   . Thus there exist $\{x_n\}, \{y_n\} \in M$ so that

$$\begin{aligned}\bar{d}([\{x_n\}], [\{y_n\}]) &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} d'(F(x_n), F(y_n)) \\ &= d'(\lim_{n\to\infty} F(x_n), \lim_{n\to\infty} F(y_n)) \\ &= d'(\phi([\{x_n\}]), \phi([\{y_n\}])\end{aligned}$$

   Hence we have that $\bar{d}([\{x_n\}], [\{y_n\}]) = d'(\phi([\{x_n\}]), \phi([\{y_n\}])$, and we say that ϕ is an isometry on $\bar{M}$ to M'.

6. Problem 1.2.18, pg. II-303.

   (a) Show that

$$d(x, y) = \frac{|x - y|}{\sqrt{(1 + x^2)(1 + y^2)}}, \tag{1}$$

   defines a metric on $\mathbb{R}$.

   Solution. We begin by showing that $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y$ and that $d(x, y) = d(y, x)$ trivially. Hence we must show that (1) satisfies the triangle inequality. To show this, we let $x, y, z \in \mathbb{R}$, and we will make use of the two basic facts that follow

$$(1 + xy)^2 \leq (1 + x^2)(1 + y^2), \tag{2}$$
$$(x - y)(1 + z^2) = (x - z)(1 + yz) + (z - y)(1 + xz). \tag{3}$$

Then

$$
\begin{aligned}
|x-y|(1+z^2) &\leq |x-z||1+yz| + |z-y||1+xz| \quad \text{from (3)}, \\
&\leq |x-z|(1+z^2)^{\frac{1}{2}}(1+y^2)^{\frac{1}{2}} + |z-y|(1+z^2)^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}
\end{aligned}
$$

where the last inequality is obtained from Eqn (2). Dividing each side by the factor of $(1+x^2)^{\frac{1}{2}}(1+y^2)^{\frac{1}{2}}(1+z^2)$ we obtain

$$
\frac{|x-y|}{\sqrt{(1+x^2)(1+y^2)}} \leq \frac{|x-z|}{\sqrt{(1+x^2)(1+z^2)}} + \frac{|z-y|}{\sqrt{(1+z^2)(1+y^2)}} \tag{4}
$$

$\therefore d(x,y)$ defines a metric on $\mathbb{R}$.

(b) Show that $\mathbb{R}$ is not complete in this metric.

Solution. To show that $\mathbb{R}$ is not complete in this metric, I must find a sequence that is Cauchy in the metric, but whose limit does not exist in $\mathbb{R}$. Consider the sequence $a_n = 2^n$, $n \in \mathbb{N}$. Notice, WLOG let $n < m$, there $\exists N_\epsilon$ such that $m, n \geq N_\epsilon$ implies that $\frac{1}{2^n} < \epsilon/3$, and $\frac{1}{2^m} < \epsilon/3$

$$
\begin{aligned}
\frac{|2^n - 2^m|}{\sqrt{(1+2^{2n})(1+2^{2m})}} &\leq \frac{2^n|2^{m-n}-1|}{\sqrt{(1+2^{2n})(1+2^{2m})}} \\
&\leq \frac{2^n|2^{m-n}-1|}{2^{m+n}} \\
&\leq \frac{|2^{m-n}-1|}{2^m} \\
&\leq \frac{2^{m-n}}{2^m} + \frac{1}{2^m} \\
&= \frac{1}{2^n} + \frac{1}{2^m} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon
\end{aligned}
$$

Hence we have that a_n is Cauchy in this metric. However, the limit clearly does not exists in $\mathbb{R}$, since for any $a \in \mathbb{R}$ that is a candidate to be the limit of a_n, we have by the ordering of $\mathbb{R}$ that there exists N_a such that $n \geq N_a$ implies $a < 2^n$. That is, after some n, the entries in a_n will grow larger than any $x \in \mathbb{R}$, hence our limit does not exist in $\mathbb{R}$. In a more rigorous sense, I will show that if $\lim_{n\to\infty} 2^n = a$, then a does not exist in any closed ball in our metric on $\mathbb{R}$.

Let $r \in \mathbb{R}$ $B_d(r,0) = \{x : d(x,0) \leq r\}$. Thus $B_d(r,0)$ is the set of all x such that

$$\frac{|x|}{\sqrt{(1+x^2)}} \leq r \tag{5}$$

Note that from (5) we have the implication that $|x| \leq r$. Hence $B_d \subseteq B_{|\cdot|}$, and if we can show that our sequence does not converge in $|\cdot|$ to some real value, then it will not converge in our metric. To show that $\{2^n\}$ does not converge in $|\cdot|$ is trivial, so we have found our sequence which is Cauchy in our metric d, but does not converge in $\mathbb{R}$, but tends to ∞. Hence $\mathbb{R}$ is not complete with the metric d.

(c) Find the completion.

Solution. From part (b), we are led to assume that the completion of $\mathbb{R}$ will be $\mathbb{R} \cup \{\infty\}$. Note that it is not necessary to include $\{-\infty\}$ in the completion, I show that $[\{n\}] \sim [\{-n\}]$

$$\begin{aligned} \bar{d}([\{n\}], [\{-n\}]) &= \lim_{n\to\infty} d(n, -n) \\ \lim_{n\to\infty} \frac{|n-(-n)|}{\sqrt{(1+n^2)(1+(-n)^2)}} &= \lim_{n\to\infty} \frac{2|n|}{(1+n^2)} \\ &= 0 \end{aligned}$$

Now, the limit of the Cauchy sequence $\{a_n\}$ from part (b) exists in the set $\mathbb{R} \cup \{\infty\}$, therefore the metric space is complete.